On the parity of ranks of Selmer groups II

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Let

$$E : y^2 = x^3 + Ax + B \qquad (A, B \in \mathbf{Q})$$

be an elliptic curve over $\mathbf{Q}$ of conductor N. Thanks to the work of Wiles and his followers [BCDT] we know that E is modular, i.e. there exists a non-constant map $\pi : X_0(N) \longrightarrow E$ defined over $\mathbf{Q}$ and

$$L(E, s) = \sum_{n=1}^{\infty} a_n n^{-s} = L(f, s)$$

for a normalized newform $f \in S_2(\Gamma_0(N))$.

For a large class of number fields F (which includes all solvable Galois extensions of $\mathbf{Q}$), this is known to imply that the L-function $L(E/F, s)$ has a holomorphic continuation to $\mathbf{C}$ and a functional equation relating the values at s and $2 - s$. For such F, denote by

$$r_{an}(E/F) := \mathrm{ord}_{s=1} L(E/F, s)$$

the *analytic rank* of E over F.

Over an arbitrary number field F, the m-descent on E gives rise (for every integer $m \geq 1$) to the Selmer group $S(E/F, m)$ sitting in the standard exact sequence

$$0 \longrightarrow E(F) \otimes \mathbf{Z}/m\mathbf{Z} \longrightarrow S(E/F, m) \longrightarrow Ш(E/F)[m] \longrightarrow 0.$$

Fix a prime number p and put

$$S(E/F) = S_p(E/F) = \varinjlim_n S(E/F, p^n)$$
$$X(E/F) = X_p(E/F) = \varprojlim_n S(E/F, p^n)$$
$$s_p(E/F) = \mathrm{corank}_{\mathbf{Z}_p} S_p(E/F) = \mathrm{rk}_{\mathbf{Z}_p} X_p(E/F).$$

The conjecture of Birch and Swinnerton-Dyer predicts that

$$(BSD) \qquad r_{an}(E/\mathbf{Q}) \stackrel{?}{=} \mathrm{rk}_{\mathbf{Z}} E(\mathbf{Q}).$$

As in [NePl], we are interested in a rather weak consequence of (BSD) (and the conjectural finiteness of the Tate-Šafarevič group), namely the

Parity conjecture for Selmer groups: $\quad r_{an}(E/\mathbf{Q}) \stackrel{?}{\equiv} s_p(E/\mathbf{Q}) \pmod 2.$

Our main result is the following

Theorem A. *Let E be an elliptic curve over $\mathbf{Q}$ with good ordinary reduction at p. Then the parity conjecture*

$$r_{an}(E/\mathbf{Q}) \equiv s_p(E/\mathbf{Q}) \pmod 2$$

holds.

See [NePl] for a discussion of earlier results in this direction. Our method of proof is similar to that in [NePl]; the only difference is that we use anticyclotomic deformations instead of Hida families. A recently proved conjecture of Mazur ([Maz2], [Co], [Va2]) on non-vanishing of Heegner points in anticyclotomic $\mathbf{Z}_p$-extensions plays the role of Greenberg's conjecture assumed in [NePl].

1. Heegner points

In this section we recall the basic setup of Heegner points on E (see [Gro] for a more detailed account).

1.1 Let $K = \mathbf{Q}(\sqrt{D})$ be an imaginary quadratic field of discriminant $D < 0$. We assume that K satisfies the following "Heegner condition" of Birch [Bi]:

$$(Heeg) \qquad \text{Every prime } q|N \text{ splits in } K.$$

Under this assumption there exists an ideal $\mathcal{N} \subset \mathcal{O}_K$ such that $\mathcal{O}_K/\mathcal{N} \xrightarrow{\sim} \mathbf{Z}/N\mathbf{Z}$ (of course, $\mathcal{N}$ is not unique; we choose one).

1.2 For every integer $c \geq 1$ denote by $\mathcal{O}_c = \mathbf{Z} + c\mathcal{O}_K$ the unique order of O_K of conductor c. If $(c, N) = 1$, then

$$\mathcal{N}_c := \mathcal{O}_c \cap \mathcal{N}$$

is an invertible ideal in $\mathcal{O}_c$ satisfying $\mathcal{O}_c/\mathcal{N}_c \xrightarrow{\sim} \mathbf{Z}/N\mathbf{Z}$ (and hence also $\mathcal{N}_c^{-1}/\mathcal{O}_c \xrightarrow{\sim} \mathbf{Z}/N\mathbf{Z}$, since $\mathcal{N}_c$ is invertible). The cyclic N-isogeny

$$[\mathbf{C}/\mathcal{O}_c \longrightarrow \mathbf{C}/\mathcal{N}_c^{-1}]$$

induced by the identity map on $\mathbf{C}$ defines a non-cuspidal point on the modular curve $X_0(N)$, which is defined over H_c, the ring class field of conductor c over K. The image of this point under the fixed modular parametrization $\pi : X_0(N) \longrightarrow E$ will be denoted by $\overline{x}_c \in E(H_c)$ – this is a *Heegner point of conductor* c *on* E.

1.3 From now on, we consider only conductors of the form $c = p^n$ for a fixed prime number p. The field $H_{p^\infty} = \bigcup H_{p^n}$ then contains the *anticyclotomic* $\mathbf{Z}_p$*-extension* K_∞ of K, which can be characterized by the following properties: $K_\infty = \bigcup K_n$, $G(K_n/K) \xrightarrow{\sim} \mathbf{Z}/p^n\mathbf{Z}$, $K_n/\mathbf{Q}$ is a Galois extension with $G(K_n/\mathbf{Q}) \xrightarrow{\sim} D_{2p^n}$ (the dihedral group). There is $n_0 \geq 0$ such that

$$H_{p^{n+1}} \cap K_\infty = K_{n+n_0} \qquad (n \geq 0).$$

We use the following standard notation:

$$\Gamma = G(K_\infty/K), \qquad \Gamma_n = G(K_\infty/K_n), \qquad \Lambda = \mathbf{Z}_p[[\Gamma]].$$

Fix an isomorphism $\Gamma \xrightarrow{\sim} \mathbf{Z}_p$ (i.e. fix a topological generator γ of Γ).

From now on, we assume that the following condition of "good reduction" is satisfied:

$$(G) \qquad p \nmid N.$$

This assumption implies that the Heegner points

$$\overline{x}_{p^{n+1}} \in E(H_{p^{n+1}})$$

are defined. We put

$$x_{n+n_0} := \mathrm{Tr}_{H_{p^{n+1}}/K_{n+n_0}}(\overline{x}_{p^{n+1}}) = \mathrm{Tr}_{H_{p^\infty}/K_\infty}(\overline{x}_{p^{n+1}}) \in E(K_{n+n_0}) \qquad (n \geq 0).$$

1.4 In his lecture at ICM 1983, Mazur formulated (among others) the following

Conjecture [Ma2]. $(\exists\, n \geq n_0) \quad x_n \notin E(K_n)_{\mathrm{tors}}$.

This conjecture was recently proved by Vatsal [Va2] under the assumptions that $|D|$ is prime and p does not divide the class number of K, and by Cornut [Co] under the assumption (G). Both [Co] and [Va2] build upon an earlier work [Va1] of Vatsal.

1.5 The fundamental distribution relation for Heegner points ([PR], Lemma 2, p.432) states that

$$\mathrm{Tr}_{K_{n+1}/K_n}(x_{n+1}) = a_p x_n - x_{n-1} \qquad\qquad (n \geq n_0 + 1).$$

The assumption (G) is equivalent to E having good reduction at p. From now on, we assume in addition that E has *ordinary* reduction at p, which is equivalent to

$$(Ord) \qquad\qquad p \nmid a_p.$$

This assumption implies that the local Euler factor of E at p factorizes as

$$1 - a_p X + pX^2 = (1 - \alpha X)(1 - \beta X),$$

where $\alpha, \beta \in \mathbf{Z}_p$ satisfy $\mathrm{ord}_p(\alpha) = 0$, $\mathrm{ord}_p(\beta) = 1$.

Define, for $n \geq n_0 + 1$,

$$y_n = x_n \otimes \alpha^{1-n} - x_{n-1} \otimes \alpha^{-n} \in E(K_n) \otimes \mathbf{Z}_p.$$

Then

$$\mathrm{Tr}_{K_{n+1}/K_n}(y_{n+1}) = y_n \qquad\qquad (n \geq n_0 + 1),$$

i.e. $y = (y_n)$ is an element of the projective limit

$$\varprojlim_{n > n_0} (E(K_n) \otimes \mathbf{Z}_p) = \varprojlim_n (E(K_n) \otimes \mathbf{Z}_p) \subseteq \varprojlim_n X(E/K_n) =: X_\infty.$$

We shall also be interested in the inductive limit

$$S_\infty = \varinjlim_n S(E/K_n).$$

Both X_∞ and S_∞ are Λ-modules (of finite and co-finite type, respectively).

2. Iwasawa theory

In this section we recall basic results of Iwasawa theory of elliptic curves relating the Λ-modules X_∞ and S_∞ to Selmer groups over the fields K_n. The assumptions (Heeg), (G) and (Ord) are in force.

Lemma 2.1. (i) $(\forall n \geq 0)$ *the canonical map*

$$S(E/K_n) \longrightarrow (S_\infty)^{\Gamma_n}$$

has finite kernel and cokernel.

(ii) *There is an isomorphism of Λ-modules (of finite type)*

$$X_\infty \xrightarrow{\sim} \mathrm{Hom}_\Lambda(\widehat{S_\infty}, \Lambda)$$

(where $\widehat{M}$ denotes the Pontryagin dual of M), hence $\mathrm{rk}_\Lambda(X_\infty) = \mathrm{corank}_\Lambda(S_\infty)$.

(iii) $(\forall n \geq 0)$ *the canonical map*

$$(X_\infty)_{\Gamma_n} \longrightarrow X(E/K_n)$$

has finite kernel.
(iv) $X_\infty \overset{\sim}{\longrightarrow} \Lambda^r$ *for some* $r \geq 0$.
(v) $E(K_\infty)_{\mathrm{tors}}$ *is finite.*

Proof. (i) [Man], Thm. 4.5 or [Maz1], Prop. 6.4 (note that [Man], Lemma 4.6 eliminates the need for the second assumption in [Maz1], 6.1; however, the latter is satisfied in our situation, because of (Heeg)). (ii) [PR], Lemma 5, p. 417. (iii) This follows from (i) and (ii) (cf. [PR], Lemma 4, p. 415). (iv) The R.H.S. in (ii) is reflexive, hence free over Λ. (v) [NeSc], 2.2.

Lemma 2.2. $y \neq 0$ *in* X_∞.

Proof. If $y_n = 0$ in $E(K_n) \otimes \mathbf{Z}_p$ for all $n > n_0$, then

$$x_n \otimes \alpha = x_{n-1} \otimes 1$$

in $E(K_n) \otimes \mathbf{Q}_p$ for all $n > n_0$. As both $x_n \otimes 1$ and $x_{n-1} \otimes 1$ lie in $E(K_n) \otimes \mathbf{Q}$, but $\alpha \notin \mathbf{Q}$, we must have $x_n \otimes 1 = 0$ in $E(K_n) \otimes \mathbf{Q}$ for all $n > n_0$, which contradicts (now proven) Mazur's conjecture 1.4.

Lemma 2.3. $X_\infty \overset{\sim}{\longrightarrow} \Lambda$.

Proof. (Note that the implication [Mazur's conjecture $\Longrightarrow X_\infty \overset{\sim}{\longrightarrow} \Lambda$] was proved under more restrictive assumptions by Bertolini [Be]).
By Lemma 2.2 there is an exact sequence of Λ-modules of finite type

$$0 \longrightarrow \Lambda y \longrightarrow X_\infty \longrightarrow X_\infty/\Lambda y \longrightarrow 0.$$

For every *surjective* character

$$\chi : \Gamma \longrightarrow \mu_{p^n}$$

we put $c(\chi) = n$ and denote by

$$e_\chi = \frac{1}{p^n} \sum_{g \in \Gamma/\Gamma_n} \chi(g)^{-1} g \in \mathbf{Q}_p(\mu_{p^n})[\Gamma/\Gamma_n]$$

the corresponding idempotent. Denote by A the (finite) set of integers $n \geq 1$ such that

$$\frac{\omega_n}{\omega_{n-1}} \in \mathrm{Supp}_\Lambda((X_\infty/\Lambda y)_{\mathrm{tors}})$$

(where $\omega_n = \gamma^{p^n} - 1$, as usual). If $c(\chi) \notin A$, then

$$e_\chi \left(\mathbf{Q}_p(\mu_{p^n}) \cdot y \bmod \omega_n X_\infty\right) \neq 0$$

in $(X_\infty)_{\Gamma_n} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p(\mu_{p^n})$, hence also

$$e_\chi \left(\mathbf{Q}_p(\mu_{p^n}) \cdot y_n\right) \neq 0$$

in $X(E/K_n) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p(\mu_{p^n})$, by Lemma 2.1(iii).

According to a mild generalization of the main result of [Be-Da],

$$e_\chi \left(\mathbf{Q}_p(\mu_{p^n}) \cdot y_n\right) \neq 0 \Longrightarrow e_\chi \left(\mathbf{Q}_p(\mu_{p^n}) \cdot y_n\right) = e_\chi \left(X(E/K_n) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p(\mu_{p^n})\right).$$

Putting everything together and again appealing to Lemma 2.1(iii), we see that

$$\mathrm{rk}_{\mathbf{Z}_p}((X_\infty)_{\Gamma_n}) \leq p^n + O(1),$$

hence $r \leq 1$ in Lemma 2.1(iv). However, $r \neq 0$ by Lemma 2.2.

2.4 Recall that finite Λ-modules are also called *pseudo-null*. Denote by $({}_\Lambda\mathrm{Mod})$ the category of all Λ-modules and by $({}_\Lambda\mathrm{Mod})/(ps - null)$ the category obtained from $({}_\Lambda\mathrm{Mod})$ by inverting all morphisms which have pseudo-null both kernel and cokernel. This is again an abelian category.

Lemma 2.5. *In $({}_\Lambda\mathrm{Mod})/(ps-null)$ there is an exact sequence*

$$0 \longrightarrow Y \oplus Y \oplus Z \longrightarrow \widehat{S_\infty} \longrightarrow \Lambda \longrightarrow 0$$

and an isomorphism

$$Z \xrightarrow{\sim} \bigoplus_{i=1}^{k} (\Lambda/p^{m_i}\Lambda).$$

Proof. Combining Lemma 2.3 with Lemma 2.1(ii) we get an exact sequence in $({}_\Lambda\mathrm{Mod})/(ps-null)$

$$0 \longrightarrow (\widehat{S_\infty})_{\mathrm{tors}} \longrightarrow \widehat{S_\infty} \longrightarrow \Lambda \longrightarrow 0.$$

The duality results of [Ne] imply that

$$(\widehat{S_\infty})_{\mathrm{tors}} \xrightarrow{\sim} Y \oplus Y \oplus Z$$

in $({}_\Lambda\mathrm{Mod})/(ps-null)$, with Z as in the statement of the Lemma (in fact, the condition (Heeg) implies that Z itself is pseudo-null, at least if $p > 2$; however, we do not need this fact).

3. Main Results

Theorem B. *Under the assumptions (Heeg) and $p \not| Na_p$ we have*

$$s_p(E/K) \equiv 1 \equiv r_{an}(E/K) \pmod 2.$$

Proof. First of all, (Heeg) implies that $r_{an}(E/K)$ is odd. Lemma 2.5 together with Lemma 2.1(i) give an exact sequence

$$0 \longrightarrow (Y_\Gamma \oplus Y_\Gamma) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \longrightarrow \mathrm{Hom}_{\mathbf{Z}_p}(X(E/K), \mathbf{Q}_p) \longrightarrow \mathbf{Q}_p \longrightarrow 0,$$

hence

$$s_p(E/K) = 1 + 2\,\mathrm{rk}_{\mathbf{Z}_p}(Y_\Gamma) \equiv 1 \pmod 2.$$

Theorem A. *Let E be an elliptic curve over $\mathbf{Q}$ with good ordinary reduction at p. Then*

$$r_{an}(E/\mathbf{Q}) \equiv s_p(E/\mathbf{Q}) \pmod 2.$$

Proof. For every $K = \mathbf{Q}(\sqrt{D})$ satisfying (Heeg) we have

$$s_p(E/K) = s_p(E/\mathbf{Q}) + s_p(E_D/\mathbf{Q})$$
$$r_{an}(E/K) = r_{an}(E/\mathbf{Q}) + r_{an}(E_D/\mathbf{Q}),$$

where

$$E_D : Dy^2 = x^3 + Ax + B$$

is the quadratic twist of E over K. We distinguish two cases:

(I) $r_{an}(E/\mathbf{Q})$ is odd.

According to [Wa] there exists K satisfying (Heeg) such that

$$r_{an}(E_D/\mathbf{Q}) = 0.$$

Results of Kolyvagin [Ko] then imply $s_p(E_D/\mathbf{Q})=0$, hence

$$s_p(E/\mathbf{Q})=s_p(E/K)\equiv r_{an}(E/K)=r_{an}(E/\mathbf{Q})\pmod 2$$

by Theorem B.

(II) $r_{an}(E/\mathbf{Q})$ is even.

Choose any K satisfying (Heeg) and $p\not| D$. Applying the result of Case (I) to E_D (which has good ordinary reduction at p), we obtain

$$s_p(E_D/\mathbf{Q})\equiv r_{an}(E_D/\mathbf{Q})\pmod 2,$$

hence

$$s_p(E/\mathbf{Q})\equiv 1-s_p(E_D/\mathbf{Q})\equiv 1-r_{an}(E_D/\mathbf{Q})\equiv r_{an}(E/\mathbf{Q})\pmod 2,$$

again using Theorem B.

4. Higher dimensional quotients of $J_0(N)$

Results of Section 3 can be generalized as follows.

4.1 Let $f=\sum_{n\geq 1}a_nq^n\in S_2(\Gamma_0(N))$ be a normalized ($a_1=1$) newform. Put $F(f)=\mathbf{Q}(a_1,a_2,\cdots)$; this is a totally real number field. Let A be a quotient abelian variety of $J_0(N)$ corresponding to f; it has dimension $[F(f):\mathbf{Q}]$, is defined over $\mathbf{Q}$ and is unique up to isogeny. One has an embedding $\iota: F(f)\hookrightarrow \mathrm{End}(A)\otimes\mathbf{Q}$; for each $n\geq 1$, $\iota(a_n)$ is induced by the Hecke correspondence $T(n): J_0(N)\longrightarrow J_0(N)$ (with respect to the Albanese functoriality).

4.2 For each prime number p, $V_p(A)=T_p(A)\otimes_{\mathbf{Z}_p}\mathbf{Q}_p$ is a free module of rank 2 over $F(f)\otimes_{\mathbf{Q}}\mathbf{Q}_p=\bigoplus_{\mathfrak{p}|p}F(f)_{\mathfrak{p}}$. Fix a prime $\mathfrak{p}$ above p in $F(f)$; then the $F(f)_{\mathfrak{p}}$-component $V_{\mathfrak{p}}(A)$ of $V_p(A)$ defines a two-dimensional Galois representation

$$G_{\mathbf{Q}}=G(\overline{\mathbf{Q}}/\mathbf{Q})\longrightarrow \mathrm{Aut}_{F(f)_{\mathfrak{p}}}(V_{\mathfrak{p}}(A))\xrightarrow{\sim} GL_2(F(f)_{\mathfrak{p}})$$

which is unramified outside Np and satisfies

$$\det(1-X\cdot Fr_{\mathrm{geom}}(\ell)|V_{\mathfrak{p}}(A))=1-a_\ell\ell^{-1}X+\ell^{-1}X^2$$

for all prime numbers $\ell\not| Np$.

4.3 Let $\mathcal{O}_{\mathfrak{p}}$ be the ring of integers of $F(f)_{\mathfrak{p}}$. For every number field F, denote by $S_{\mathfrak{p}}(A/F)$ the $\mathfrak{p}$-primary part of the Selmer group $S_p(A/F)=\varinjlim_n S(A/F,p^n)$. This is an $\mathcal{O}_{\mathfrak{p}}$-module of cofinite type; denote by $s_{\mathfrak{p}}(A/F)$ its $\mathcal{O}_{\mathfrak{p}}$-corank. Note that $S_{\mathfrak{p}}(A/F)$ depends only on the $G(\overline{\mathbf{Q}}/F)$-module $V_{\mathfrak{p}}(A)/T_{\mathfrak{p}}(A)$, where $T_{\mathfrak{p}}(A)$ is the $\mathcal{O}_{\mathfrak{p}}$-component of $T_p(A)$, as it coincides with a Bloch-Kato Selmer group

$$S_{\mathfrak{p}}(A/F)=H^1_f(F,V_{\mathfrak{p}}(A)/T_{\mathfrak{p}}(A)).$$

The role of $r_{an}(E/F)$ is played by the order of vanishing

$$r_{an}(f,F)=\mathrm{ord}_{s=1}L(f\otimes F,s).$$

Under the assumptions (Heeg), (G) and

$$(Ord')\qquad\qquad \mathrm{ord}_{\mathfrak{p}}(a_p)=0,$$

the arguments in Section 2 go through for the $\mathcal{O}_{\mathfrak{p}}$-modules $S_{\mathfrak{p}}(A/K_n)$ (Mazur's control Theorem has to be replaced by a purely cohomological "control theorem"; see [Gre]). Similarly, all of the arguments of Section 3 work, if we replace the reference [Ko] by [KoLo]. The final results are the following.

Theorem B'. *Under the assumptions (Heeg), $p \nmid N$ and $\mathrm{ord}_{\mathfrak{p}}(a_p) = 0$ we have*

$$s_{\mathfrak{p}}(A/K) \equiv 1 \equiv r_{an}(f, K) \pmod{2}.$$

Theorem A'. *Assume that $p \nmid N$ and $\mathrm{ord}_{\mathfrak{p}}(a_p) = 0$. Then*

$$\mathrm{ord}_{s=1} L(f, s) \equiv s_{\mathfrak{p}}(A/\mathbf{Q}) \pmod{2}.$$

More general results can be deduced by applying the techniques of [NePl]. This will be discussed in a separate publication.

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